UNIRULED SURFACES OF GENERAL TYPE

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ABSTRACT. We give a systematic construction of uniruled surfaces in positive characteristic. Using this construction, we find surfaces of general type with non-trivial global vector fields, surfaces with arbitrarily non-reduced Picard schemes as well as surfaces with inseparable canonical maps. In particular, we show that some previously known pathologies are not sporadic but exist in abundance.

CONTENTS

INTRODUCTION

In the beginning of the 20th century, the Italian school established a coarse classification of complex surfaces of special type, the so-called Enriques classification. Since then, it has been clarified and refined by Kodaira, Šafarevič and many others. In a series of three papers in the 1970s, Bombieri and Mumford [BM] extended this classification to positive characteristic. A major new feature is the existence of fibrations over curves with singular geometric generic fibres. For surfaces of special type, only quasi-elliptic surfaces, which exist in characteristic 2 and 3 only, have to be considered. By definition, a quasi-elliptic surface is a surface that admits a fibration over a curve such that the geometric generic fibre is a singular rational curve of arithmetic genus 1. In particular, a quasi-elliptic surface is (inseparably) uniruled.

But what about the classification of complex surfaces of general type? A full classification seems out of range at the moment. However, one would like to have bounds on their invariants, to understand the behaviour of their pluricanonical systems, and to classify surfaces with low invariants. Famous results in this direction are the Bogomolov–Miyaoka–Yau inequality $c_1^2 \leq 9\chi$ and Bombieri's analysis of the pluricanonical systems, just to name two. The situation in positive characteristic is more complicated: many equalities and inequalities, which easily follow from Hodge theory, do not hold simply because the Frölicher spectral sequence from Hodge to de Rham cohomology does not degenerate at

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E_1-level in general. The Bogomolov–Miyaoka–Yau inequality is known to fail [Sz, Section 3.4.1]. On the other hand, Ekedahl [E2] and Shepherd-Barron [S-B] extended Bombieri's results on pluricanonical maps to positive characteristic. Again, the exceptions occur mainly in small characteristics and (inseparably) uniruled surfaces are responsable for many unexpected phenomena. Thus, in order to understand failures of classical theorems about surfaces in positive characteristic, it is indispensable to understand uniruled surfaces. And as a first step, one should have at least a large supply of examples to study.

In this paper, we present a systematic construction of surfaces that are uniruled or birationally dominated by Abelian surfaces. This construction is inspired by the structure result of quasi-hyperelliptic surfaces [BM, Theorem 1] and Schröer's construction [Sch] of unirational K3 surfaces.

To obtain our surfaces, we need two curves C, F and rational p-closed vector fields δ_C, δ_F on them. Resolving the singularities of the quotient of $C \times F$ by $\delta_C + \delta_F$, we obtain a surface X which is birationally dominated by $C \times F$. In particular, if at least one of the curves is rational, the surface X is uniruled. If both curves are rational then X is even unirational. The surface X comes with fibrations over $C^{(-1)}$ and $F^{(-1)}$, and both fibrations are usually not generically smooth.

We will restrict us mostly to characteristic 2. This is mainly because we expect to find more pathologies in this characteristic and partly to simplify our exposition. For example, the restriction to characteristic 2 allows an analysis of the slope spectral sequence and the spectral sequence from Hodge to de Rham cohomology. However, before stating these results we first discuss the examples we have found.

For curves of general type, i.e. of genus at least 2, the canonical map is always a separable morphism, which is either of degree 2 onto $\mathbb{P}^1$ or defines an embedding. Already in characteristic zero, the situation for surfaces is more complicated and we refer to [BHPV, Section VII.7] for an introduction as well as references. Here we show that the canonical map can become inseparable and that this is not a sporadic phenomenon:

Theorem 7.6. *In characteristic 2 there exist unbounded families of unirational surfaces of general type whose canonical maps are inseparable morphisms onto rational surfaces.*

We note that it follows from Shepherd-Barron's results [S-B, Theorem 27] that $|3K_X|$ of a surface of general type defines a birational morphism provided that c_1^2 and χ are sufficiently large. Moreover, if X does not possess a pencil of curves of arithmetic genus 2 then already $|2K_X|$ defines a birational morphism if c_1^2 and χ are sufficiently large. However, we can arrange the surfaces of Theorem 7.6 not to possess pencils of curves of small arithmetic genus. Hence the inseparability of the canonical map is not related to the existence of special fibrations of low genus.

It is already known for some time that the Bogomolov–Miyaoka–Yau inequality $c_1^2 \leq 9\chi$ may fail in positive characteristic [Sz, Section 3.4.1]. Here we present a surface violating this inequality with $\chi = 1$, which is the smallest value possible for χ in characteristic zero.

Theorem 6.1. *In characteristic 2 there exist surfaces of general type with $\chi = 1$ and $c_1^2 = 14$.*

The number of isolated (-2)-curves on a minimal surface of general type is bounded above by $\frac{1}{9}(3c_2 - c_1^2)$ by a theorem of Miyaoka. Also this is known to fail in positive characteristic and Shepherd-Barron [S-B2, Theorem 4.1] has shown that if the number of (-2)-curves exceeds $c_1^2 + \frac{1}{2}c_2$ then the surface is uniruled. However, usually there is a gap between these two bounds and we show that this gap is populated by uniruled as well at non-uniruled surfaces. In particular, Shepherd-Barron's bound is not sharp:

Theorem 6.3. *There exist minimal surfaces of general type in characteristic 2 violating Miyaoka's bound on (-2)-curves that do not reach Shepherd-Barron's bound. There exist uniruled as well as non-uniruled such surfaces.*

Since a group scheme over a field of positive characteristic may be non-reduced one has to distinguish between the Picard variety and the Picard scheme of a variety. Examples of surfaces with non-reduced Picard schemes fields are known, e.g. [Ig2]. However, one could ask whether there are bounds on the non-reducedness, e.g. one could ask whether the dimension $h^1(\mathcal{O}_X)$ of the tangent space to the Picard scheme $\mathrm{Pic}(X)$ is bounded, say, in terms of the dimension $\frac{1}{2}b_1(X)$ of the Picard scheme. This is not the case:

Theorem 7.1. *Given an integer $q \geq 2$, there exists a family $\{X_i\}_{i\in\mathbb{N}}$ of uniruled surfaces of general type in characteristic 2 all having the same Picard variety of dimension q such that*

$$h^{01}(X_i) \,=\, h^1(\mathcal{O}_{X_i}) \,\to\, \infty \text{ as } i \to \infty$$

Thus, the Picard scheme can get arbitrarily non-reduced, even when fixing the Picard variety.

It follows from Hodge theory that all global 1-forms on a complex projective manifold are pull-backs of global 1-forms from its Albanese variety via the Albanese map. On the other hand, Igusa [Ig2] gave an example of a surface with $h^0(\Omega^1_X)$ strictly larger than $\frac{1}{2}b_1(X) = \dim \mathrm{Alb}(X) = h^0(\Omega^1_{\mathrm{Alb}(X)})$ and

Theorem 7.2. *Given an integer $q \geq 2$, there exists a family $\{X_i\}_{i\in\mathbb{N}}$ of uniruled surfaces of general type in characteristic 2 all having the same Albanese variety of dimension q such that*

$$h^{10}(X_i) \,=\, h^0(\Omega^1_{X_i}) \,\to\, \infty \text{ as } i \to \infty\,.$$

Combining Theorem 7.1 and Theorem 7.2 we can even produce families $\{X_i\}_{i\in\mathbb{N}}$ of uniruled surfaces of general type with fixed Albanese variety and $h^{10}(X_i) - h^{01}(X_i)$ tending to infinity as i tends to infinity. Thus, we can also violate the Hodge symmetry "$h^{10} = h^{01}$" as much as we want - even when fixing the first Betti number.

The automorphism group of a surface of general type over the complex numbers is finite. Hence its Lie algebra, which can be identified with the space of global holomorphic vector fields, is trivial. In particular, a surface of general type over the complex numbers does not possess non-trivial global vector fields. However, surfaces of general type with non-trivial global vector fields in positive characteristic have been constructed by Lang [La], Shepherd-Barron [S-B2] and others. Again, we are able to obtain this phenomenon in unbounded families.

Theorem 7.5. *In characteristic 2, there exist unbounded families of surfaces of general such that each member of this family possesses non-trivial global vector fields. Moreover, we can find such families in which every member is uniruled, resp. not uniruled.*

In characteristic 2, our construction has the nice feature that it is possible to compute invariants that are usually difficult to determine. Since we can easily produce surfaces that do not lift to characteristic zero, even over a ramified extension of the Witt ring, these computations may be an interesting testing ground for general conjectures about surfaces.

More precisely, we determine the Betti and Hodge numbers and analyse the spectral sequences related to Hodge and de Rham–Witt cohomology. It turns out that the spectral sequence from Hodge to de Rham cohomology may or may not degenerate at E_1-level (we present examples for both cases), whereas the slope spectral sequence from de Rham–Witt to crystalline cohomology usually does not

degenerate at E_1-level. Also, we verify the Artin–Tate conjecture for our surfaces and show that they possess an Artin invariant, analogous to the one defined by Artin [Ar] for supersingular K3 surfaces.

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1. RATIONAL VECTOR FIELDS ON CURVES

This section deals with rational vector fields on smooth curves over fields of characteristic $p > 0$. We need a supply of such vector fields for our constructions. But since a smooth curve of genus $g \geq 2$ does not possess any non-trivial vector fields and the existing vector fields on curves of genus 0 and 1 are usually not interesting for us, we have to work with rational vector fields from the very beginning.

For a vector field δ_C on a curve C we denote by (δ_C) its divisor, by $(\delta_C)_0$ its divisor of zeros and by $(\delta_C)_\infty$ its divisor of poles. Thus, $(\delta_C) = (\delta_C)_0 - (\delta_C)_\infty$.

We recall that a rational vector field δ is called p*-closed*, if $\delta^{[p]} = f \cdot \delta$ for some rational function f on C. If $f = 0$ the vector field is called *additive*, whereas it is called *multiplicative* if $f = 1$.

Rational curves. Let x be a coordinate on $\mathbb{P}^1$. We consider the following rational vector fields, which are easily seen to be additive in characteristic 2.

$$\delta_1 := (x^{-4} + x^{-2}) D_x \tag{1}$$

$$\delta_2 := (x^{-2} + x^{4}) D_x \tag{2}$$

The zeros of both vector fields are of order 2. The vector field δ_1 has a pole of order 4 at $x = 0$, whereas δ_2 has poles of order 2 at $x = 0$ and $x = \infty$.

More generally, we choose pairwise distinct elements $a_1, ..., a_n$ and $b_1, ..., b_n$ of the ground field k. Then the rational vector field

$$\delta'_n := \prod_{i=1}^{n} (x - a_i)^2 \, (x - b_i)^{-2} \, D_x \tag{3}$$

has only zeros and poles of order 2 and is additive in characteristic 2. More precisely, its poles lie at $x = b_i$ and its $n + 1$ zeros are given by $x = a_i$ and $x = \infty$.

Elliptic curves. The Deuring normal form of an elliptic curve in characteristic 2 is given by

$$y^2 + \alpha x y + y = x^3 ,$$

where $\alpha \in k$ satisfies $\alpha^3 \neq 1$, cf. [Si, Appendix A]. We denote by E_α the closure of this affine curve in $\mathbb{P}^2$, which is a smooth elliptic curve with j-invariant $\alpha^{12}/(\alpha^3 - 1)$. There exists a p-closed regular vector field on E_α of additive type, i.e. the curve is supersingular, if and only if $\alpha = 0$.

The rational vector field

$$\delta_{\alpha,a,b} := (a + b \cdot x) \cdot ((1 + \alpha x) \, D_x + (\alpha y + x^2) \, D_y) . \tag{4}$$

on $\mathbb{P}^2$ descends to a rational vector field on E_α. This rational vector field is additive iff $a\alpha + b = 0$ and multiplicative iff $a\alpha + b = 1$. In particular, if $\alpha \neq 0$, i.e. if E_α is not supersingular, the rational vector field $\delta_{\alpha,1,\alpha}$ on E_α is additive and has one zero and one pole of order 2.

We would like to mention how we found these vector fields: If we consider the affine curve in Deuring normal form as lying inside $\mathbb{A}^1 \times \mathbb{A}^1$ then its projective closure F_α in $\mathbb{P}^1 \times \mathbb{P}^1$ is a singular elliptic curve with a cusp at infinity. The vector space $H^0(F_\alpha, \Theta_{F_\alpha})$ is 2-dimensional, where Θ_{F_α} denotes the dual of the sheaf of Kähler differentials. This space is explicitly described by (4) and a rational vector field extends to a regular vector field on its normalisation E_α iff $a = 0$.

Hyperelliptic curves in characteristic 2. A smooth curve C of genus $g \geq 2$ is called *hyperelliptic* if it admits a separable morphism $\varphi : C \to \mathbb{P}^1$ of degree 2. By [Bh, Section 1] we may assume that φ is branched over $g+1$ distinct points $P_1, ..., P_{g+1}$. In particular, all higher ramification groups G_i vanish already for $i \geq 2$, which is the lowest value possible in presence of wild ramification, cf. [Se, Chapitre IV]. We let x be a parameter on $\mathbb{P}^1$. By [Bh, Proposition 1.5], there exists a polynomial $g(x)$ of degree $g+1$ that does not vanish in any of the P_i such that C is given over $\mathbb{A}^1 := \mathbb{P}^1 - \{x = \infty\}$ by the equation

$$z^2 \,+\, f(x)\,z \,+\, f(x)\,g(x) \,=\, 0, \text{ where } f(x) \,:=\, \prod_{i=1}^{g+1}(x - \alpha_i)\,,$$

where the α_i correspond to the P_i. Straight forward local calculations give the divisor

$$(\partial/\partial x) \quad = \quad 2\,P'_\infty \,+\, 2\,P''_\infty \,+\, \sum_{i=1}^{g+1}(-2)\,P'_i\,. \tag{5}$$

Here, P'_i is the unique point of C lying above P_i, and P'_∞, P''_∞ lie above $x = \infty$.

Artin–Schreier extensions of $\mathbb{P}^1$. We consider the Artin–Schreier extension

$$z^p \,-\, z \,=\, x^{hp-1}$$

of the affine line in characteristic p. Its projective closure is a curve C of genus $g = 1-p+\frac{1}{2}p(p-1)h$ together with a morphism $C \to \mathbb{P}^1$, which is wildly ramified at infinity. Pulling back x we obtain

$$(\partial/\partial x) \,=\, p(h(p-1)-2)\,P_\infty\,. \tag{6}$$

By the Deuring-Šafarevič-formula [Cr, Corollary 1.8], the p-rank of such a curve is equal to zero. In particular, a line bundle $\mathcal{L}$ on C with $\mathcal{L}^{\otimes p} \cong \mathcal{O}_C$ is trivial.

2. Singular vector fields on surfaces

In this section we consider rational p-closed vector fields on surfaces with small multiplicity. In characteristic $p = 2$, this allows us to determine the the singularities of the quotient of a surface by these vector fields.

First, we recall some well-known facts from [RS, Section §1]. On a smooth surface S, we can write a rational vector field around a point P in local coordinates as

$$\delta \,=\, h(x,y)\,\left(\, f(x,y)\,\frac{\partial}{\partial x} \,+\, g(x,y)\,\frac{\partial}{\partial y}\,\right)$$

where h is a rational function and where f and g are regular functions around the point P such that the ideal $\mathcal{I} := (f, g)$ generated by f and g has height at least 2.

If $\mathcal{I}$ is not the unit ideal, the vector field is said to have an *isolated singularity* at P. The *multiplicity* of δ in P is the dimension of the k-vector space $\mathcal{O}_P/\mathcal{I}$. In case the rational vector field has no isolated singularities the vector field is said to have *only divisorial singularities*. Locally around P, the function h defines a divisor and all these functions at all points of S define a divisor, the *divisor* (δ) of the rational vector field.

If δ is a p-closed vector field on S then we can form the quotient S/δ, which is a normal surface. Its isolated singularities lie below those points of S where the vector field δ has an isolated singularity.

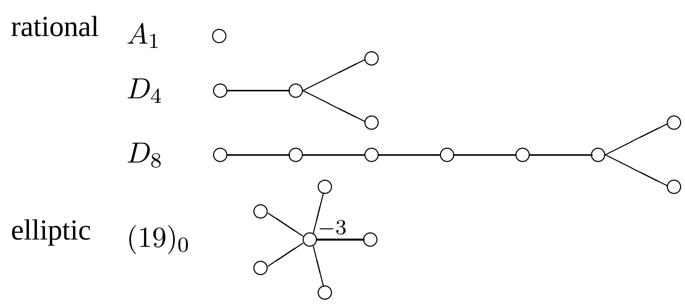
If we blow up S at an isolated singularity of δ, then we obtain an induced vector field on the blow-up. One would like to find a finite sequence of blow-ups such that the induced vector field $\tilde{\delta}$ on the blow-up $\tilde{S}$ is a rational vector field with only divisorial singularities. This would yield a diagram

$$\begin{array}{ccc} S & \leftarrow & \tilde{S} \\ \downarrow & & \downarrow \\ S/\delta & \leftarrow & \tilde{S}/\tilde{\delta} \end{array}$$

where $\tilde{S}/\tilde{\delta}$ resolves the singularities of S/δ. In general, this is not possible. However, there is the following remarkable result from [Hi, Proposition 2.6].

Proposition 2.1 (Hirokado)**.** *The singularities of a p-closed vector field on a surface in characteristic $p = 2$ can be resolved by a sequence of blow-ups.*

We recall that a singularity on X is called *rational* (resp. *elliptic*) if $R^1 f_* \mathcal{O}_{\tilde{X}}$ is a zero-dimensional (resp. one-dimensional) vector space for one, and hence every, resolution of singularities $f : \tilde{X} \to X$. In our examples, we will need the following singularities and their dual resolution graphs: every vertex represents a rational curve, which has self-intersection number -2 unless labelled differently.



Let us now assume that the vector field locally takes the form

$$\delta \;=\; f(x)\,\frac{\partial}{\partial x} \;+\; g(y)\,\frac{\partial}{\partial y}\,, \tag{7}$$

where f and g are rational functions. From the preceding discussion it is clear that

Remark 2.2. The isolated singularities of the vector field (7) are those points where f and g both have a pole or where they both have a zero.

We now determine the type of singularity the quotient acquires if the multiplicity is small.

Proposition 2.3. *Let δ be a p-closed rational vector field in characteristic 2 as in formula (7). If δ has an isolated singularity at the origin $x = y = 0$ then the quotient acquires*

(1) *a rational singularity of type A_1 if $|\mathrm{ord}_x f| = |\mathrm{ord}_y g| = 1$,*
(2) *a rational singularity of type D_4 if $|\mathrm{ord}_x f| = |\mathrm{ord}_y g| = 2$,*
(3) *a rational singularity of type D_8 if $|\mathrm{ord}_x f| = 4$ and $|\mathrm{ord}_y g| = 2$,*
(4) *an elliptic singularity of type $(19)_0$ if $|\mathrm{ord}_x f| = |\mathrm{ord}_y g| = 4$,*

at the point lying below the origin.

PROOF. In the first case, the vector field has multiplicity 1 and it follows from [Hi, Proposition 2.2] and [Hi, Corollary 2.5] that the quotient is a Du Val singularity of type A_1.

Now suppose that $\mathrm{ord}_x f = \mathrm{ord}_y g = 2$. Then there exist regular functions $\epsilon(x) = \epsilon_0 + \epsilon_1 x + ...$ and $\eta(y) = \eta_0 + \eta_1 y + ...$ with $\epsilon_0 \neq 0$ and $\eta_0 \neq 0$ such that

$$\delta \,=\, x^2\epsilon(x)\,\frac{\partial}{\partial x} \,+\, y^2\,\eta(y)\,\frac{\partial}{\partial y}\,.$$

On the blow-up with coordinates s and y, where $x = sy$, the induced rational vector field is

$$\tilde{\delta} \,=\, y\,\left((s^2\epsilon(sy) - s\eta(y))\,\frac{\partial}{\partial s} \,+\, y\eta(y)\,\frac{\partial}{\partial y} \right)\,,$$

where $y = 0$ is the local equation of the exceptional divisor E of the blow-up.

The vector field $\tilde{\delta}$ has three isolated singularities of multiplicity 1 on E at $s = 0$, $s = \infty$ and $s = \eta_0/\epsilon_0$. By Hirokado's result quoted above, these correspond to A_1-singularities and blowing up these three isolated points the induced vector field $\tilde{\delta}$ on this blow-up $\tilde{S}$ has only divisorial singularities. This yields the resolution graph of the singularity which looks like D_4.

The exceptional divisors of the resolution of singularities $\tilde{S}/\tilde{\delta} \to S/\delta$ are dominated by the exceptional divisors of the blow-up $\tilde{S} \to S$. In particular, all exceptional divisors of the resolution of singularities are rational curves. Using [RS, §1, Proposition 1], it is easy to see that all self-intersection numbers are equal to (-2). Since the intersection of these curves with the canonical class turns out to be zero, these curves are in fact smooth rational curves. This shows that the quotient by δ is indeed a singularity of type D_4.

We leave the tedious calculations of the remaining cases to the reader. □

3. UNIRULED SURFACES

We now present our construction for uniruled surfaces. It is inspired by the classification of quasi-hyperelliptic surfaces by Bombieri and Mumford [BM, Section 2], as well as the construction of K3 surfaces via the self-product of two cuspidal rational curves by Schröer [Sch].

We consider the following data $(C,\, F,\, \delta \,:=\, \delta_C + \delta_F)$

(1) two smooth curves C and F,
(2) a p-closed rational vector field δ_C on C and
(3) a p-closed rational vector field δ_F on F, where
(4) δ_C and δ_F are either both additive or both multiplicative.

We define $S := C \times F$ and denote by δ the vector field $\delta_C + \delta_F$ on S. The vector field δ is locally of the form (7) and so Remark 2.2 as well as Proposition 2.3 apply.

A theorem of Jacobson [DG, Section II.7.2] states that

$$(\delta_C \,+ \delta_F)^{[p]} \,=\, \delta_C^{[p]} \,+\, \Lambda_p(\delta_C, \delta_F) \,+\, \delta_F^{[p]}\,,$$

where $\Lambda_p(-,-)$ is a universal expression in terms of iterated Lie brackets. Considered as vector fields on S, the Lie bracket $[\delta_C, \delta_F]$ is equal to zero and so also $\Lambda_p(\delta_C, \delta_F)$ is zero. Hence, if δ_C and δ_F are both additive (resp. multiplicative) the same is true for δ.

Definition 3.1. Given (F, C, δ) we let $S := C \times F$. We will say that X is a *surface constructed from data* (C, F, δ) if there exists a sequence of blow-ups $\tilde{S} \to S$ such that the induced vector field $\tilde{\delta}$ on $\tilde{S}$ has only divisorial singularities such that $X = \tilde{S}/\tilde{\delta}$. If F is a rational curve we will refer to X as a *uniruled surface constructed from data* (C, F, δ).

Indeed, if F is a rational curve then X is dominated by a blow-up of a ruled surface and thus X is (inseparably) uniruled. In case C and F are both rational, the surface X is (inseparably) unirational.

Given a surface X constructed from data (C, F, δ), we have a commutative diagram

$$\begin{array}{ccc} S & \leftarrow & \tilde{S} \\ \downarrow & & \downarrow \\ S/\delta & \leftarrow & X \end{array}$$

where X is a smooth surface since $\tilde{\delta}$ has only divisorial singularities. More precisely, X resolves the singularities of S/δ. The map $\tilde{S} \to X$ is a finite and purely inseparable morphism of degree p and height 1, which makes the computation of the invariants of X quite easy as we will see. We note that by Hirokado's result (see Proposition 2.1 above), the assumption on the existence of a suitable blow-up $\tilde{S} \to S$ in Defintion 3.1 is automatic when working in characteristic 2.

Numerical invariants. As first immediate consequences we have the following two results.

Proposition 3.2. *Let X be a uniruled surface constructed from data (C, F, δ).*

In the Zariski topology, X is homeomorphic to a birationally ruled surface over C. Moreover,

$$\pi_1^{\text{ét}}(X) \cong \pi_1^{\text{ét}}(C) \ \text{ and thus } \ b_1(X) = b_3(X) = 2g(C).$$

If $g(C) \geq 1$, the image B of the Albanese map of X is isomorphic to C or $C^{(-1)}$ and the Albanese variety of X is isomorphic to the Jacobian of B.

PROOF. By construction, there exists a blow-up $\tilde{S}$ of $C \times \mathbb{P}^1$ and a finite and purely inseparable morphism $\pi : \tilde{S} \to X$. The map π induces a homeomorphism in the Zariski topology. By [SGA1, Théorème IX.4.10], the étale Betti numbers and algebraic fundamental groups of X and $\tilde{S}$ coincide.

The Albanese variety of $\tilde{S}$ is the Jacobian of C. Since $\pi : \tilde{S} \to X$ factors over the Frobenius morphism $F_{\tilde{S}} : \tilde{S} \to \tilde{S}^{(-1)}$, it follows from the Albanese property that $\alpha(X)$ is a curve that is birational to C or $C^{(-1)}$. Hence, the normalisation B of $\alpha(X)$ is isomorphic to C or $C^{(-1)}$. By the Albanese property, there exists a map from $\mathrm{Alb}(X)$ to the Jacobian $\mathrm{Jac}(B)$. This yields a map from $\alpha(X)$ to B, which is an inverse to the normalisation morphism. Hence $\alpha(X)$ is isomorphic to B. Applying the Albanese property to $\mathrm{Alb}(X)$ and $\mathrm{Jac}(B)$, it follows that $\mathrm{Alb}(X)$ is isomorphic to $\mathrm{Jac}(B)$. □

Proposition 3.3. *For data $(C, F, \delta_C + \delta_F)$ the quotient $X' := (C \times F)/\delta$ is a normal surface and its singularities lie below $(\delta_C)_0 \times (\delta_F)_0$ and $(\delta_C)_\infty \times (\delta_F)_\infty$. The canonical Weil divisor $K_{X'}$ is $\mathbb{Q}$-Cartier with self-intersection number*

$$K_{X'}^2 = \frac{2}{p} \cdot (2g(C) - 2 + (p-1)d_C) \cdot (2g(F) - 2 + (p-1)d_F) \ ,$$

where d_C denotes the degree of $(\delta_C)_\infty$ and d_F denotes the degree of $(\delta_F)_\infty$.

PROOF. We already noted that X' is normal and Remark 2.2 tells us where to find the singularities of X', which are isolated points. A local computation shows that the divisor of δ is equal to

$$(\delta) = -(\delta_C)_\infty \cdot F - (\delta_F)_\infty \cdot C \,. \tag{8}$$

Outside the singular locus of X' we have an equality of Cartier divisors

$$K_S = \pi^* K_{X'} + (p-1) \cdot (\delta) \,, \tag{9}$$

where $\pi : S \to X'$ is the quotient map. The Weil divisor $K_{X'}$ corresponds to a reflexive sheaf of rank 1 and so (9) extends to an equality of Weil divisors on the whole of X'.

Moreover, being reflexive and of rank 1, the divisor $K_{X'}$ defines an element in the class group of every local ring of X'. These class groups are Abelian p-torsion groups since X' is the quotient of a

smooth variety by a p-closed derivation, cf. [Fo, Chapter IV.17]. Hence $pK_{X'}$ is locally principal, i.e. a Cartier divisor. In particular, $K_{X'}$ is a $\mathbb{Q}$-Cartier divisor.

The assertion on the self-intersection number follows from (9) and the projection formula, which we may use since we are dealing with $\mathbb{Q}$-Cartier divisors. □

The following will be useful later on. If $\pi : \tilde{S} \to X$ is a finite and purely inseparable morphism of degree p and height 1 between smooth varieties then there exists an exact sequence [E, Corollary 3.4]

$$0 \to F^*\sigma^*\Omega_{\tilde{S}/X} \to \pi^*\Omega^1_X \to \Omega^1_{\tilde{S}} \to \Omega_{\tilde{S}/X} \to 0\,, \tag{10}$$

where F is the k-linear and σ is the absolute Frobenius morphism of $\tilde{S}$.

Taking determinants and applying (9) to $\tilde{S}$ and X, we obtain

$${\Omega_{\tilde{S}/X}}^{\otimes(1-p)} \cong \omega_{\tilde{S}} \otimes \pi^*{\omega_X}^\vee \cong \mathcal{O}_{\tilde{S}}((p-1)\cdot(\widetilde{\delta}))\,, \tag{11}$$

where $\tilde{\delta}$ is the rational vector field defining π.

Singular fibrations. We will now show that our surfaces are endowed with two fibrations, both of which are usually not generically smooth. Together with the results of Section 1, we use the following proposition to construct fibrations with prescribed singularities of the geometric generic fibre.

Since regularity, and hence also normality, are not stable under inseparable field extensions, we define the *arithmetic genus* p_a of an irreducible curve C, which is defined over a possibly non-perfect ground field L, to be $1-\chi(\mathcal{O}_C)$. For such a curve, we define its *geometric genus* g to be the genus of the normalisation of $C\otimes_L \bar{L}$, where $\bar{L}$ is an algebraic closure of L.

Proposition 3.4. *Let X be a surface constructed from data $(C, F, \delta = \delta_C + \delta_F)$ and denote by d_F the degree of the divisor of poles $(\delta_F)_\infty$ of δ_F. We assume that δ_C is non-trivial.*

Then there exists a fibration $f : X \to C^{(-1)}$. Its generic fibre F'_η is a regular curve over $k(C)^p$ of arithmetic genus

$$p_a(F'_\eta) = g(F) + \tfrac{p-1}{2}\cdot d_F\,.$$

The normalisation of the geometric generic fibre is isomorphic to $F\otimes_k \overline{k(C)^p}$ and its singular points are cusps lying below the points where δ_F has a pole. More precisely, if δ_F has a pole of order m in a point $P\in F$, then the arithmetic genus of the singularity of the geometric generic fibre at the cusp lying below P is equal to $\frac{p-1}{2}\cdot m$.

PROOF. By definition, there exists a blow-up $\tilde{S}$ of $S := C\times F$ and a finite inseparable morphism $\pi : \tilde{S} \to X$ of height 1. Factoring the Frobenius morphism, we obtain a morphism $\varpi : X \to \tilde{S}^{(-1)}$. The latter surface has a morphism onto $S^{(-1)}$ and a projection onto its factor $C^{(-1)}$. Composing, we obtain a morphism f from X onto $C^{(-1)}$.

The generic fibre of the fibration of $\tilde{S}^{-1}$ over $C^{(-1)}$ is $F^{(-1)}$. Since ϖ is purely inseparable, the generic fibre of f is homeomorphic to $F^{(-1)}$. As ϖ has height 1, the normalisation of the geometric generic fibre of f is isomorphic to $F\otimes_k \overline{k(C)^p}$. The normalisation map is easily seen to be a homeomorphism, and so the singularities of the geometric generic fibre are unibranch, i.e. cusps.

The local rings of the generic fibre are localisations of the local rings of the surface X, which is normal. Thus, these rings are normal and being 1-dimensional, they are regular. Hence the generic fibre is regular.

To compute the arithmetic genus of the geometric generic fibre $A := F'_\eta \otimes_k \overline{k(C)^p}$, we use the adjunction formula $2p_a(A)-2 = K_X\cdot F'_\eta + F'^2_\eta$. Since F'_η is a fibre, its self-intersection number is zero. We assumed δ_C to be non-trivial. Hence F is not an integral curve for δ and so $\pi_*F = F'_\eta$

by [RS, §1 Proposition 1]. Then the projection formula yields $F'_\eta \cdot K_X = F \cdot \pi^* K_X$. Since we are dealing with the generic fibre we can ignore contributions coming from the exceptional divisors on $\tilde{S}$. Thus we compute the intersection numbers on the singular surface X' and obtain the asserted formula for $p_a(A)$ using formulae (8) and (9).

Performing the previous calculations locally, we see that the singularities of the (geometric) generic fibre correspond to those points where δ_F has a pole. □

Remark 3.5. This result illustrates Tate's theorem [T] that the genus change of a curve in inseparable field extensions is divisible by $\frac{p-1}{2}$.

In the analysis of hyperelliptic surfaces, one of the two (quasi-) elliptic fibrations comes from the Albanese map. Joining the proofs of Proposition 3.2 and Proposition 3.4 together we obtain

Corollary 3.6. *If F is rational and $g(C) \geq 1$, then this fibration coincides with the Stein factorisation of the Albanese morphism of X.* □

4. Hodge theory in characteristic 2

To get finer invariants, we work in characteristic 2, where some of the computations become easier. Given $\pi : \tilde{S} \to X$, there exists a unique morphism $\varpi : X \to \tilde{S}^{(-1)}$ such that $\varpi \circ \pi$ is equal to the Frobenius map of $\tilde{S}$. Since ϖ is flat of degree 2, there exists a line bundle $\mathcal{L}$ and a short exact sequence

$$0 \to \mathcal{O}_{\tilde{S}^{(-1)}} \to \varpi_* \mathcal{O}_X \to \mathcal{L}^\vee \to 0\,. \tag{12}$$

By [E2, Proposition I.1.11], the map ϖ has the structure of a torsor under the finite flat group scheme $\alpha_\mathcal{L}$. We recall that $\alpha_\mathcal{L}$ is defined to be the kernel of the Frobenius map from $\mathcal{L}$ to $\mathcal{L}^p$ where we regard these line bundles as group schemes in the flat topology on $\tilde{S}^{(-1)}$.

A nice feature is that a possibly singular $\alpha_\mathcal{L}$-torsor X over a smooth base $\tilde{S}^{(-1)}$ is automatically Gorenstein. More precisely

$$\omega_X \cong \varpi^*(\omega_{\tilde{S}^{(-1)}} \otimes \mathcal{L}^{\otimes(p-1)})\,, \tag{13}$$

where ω_X denotes the dualising sheaf of X, cf. [E2, Proposition I.1.7].

Hodge numbers. With these preliminary remarks we determine Hodge numbers.

Proposition 4.1. *Let X be a uniruled surface constructed from data $(C, F, \delta = \delta_C + \delta_F)$ in characteristic 2. We let d_C and d_F be the degrees of the divisors of poles $(\delta_C)_\infty$ and $(\delta_F)_\infty$, respectively.*

The numerical equivalence class L of the line bundle $\mathcal{L}$ is given by

$$-L \equiv \left(-\frac{d_F}{2} - 1\right) C + \left(-\frac{d_C}{2} + g(C) - 1\right) F + \sum_i a_i\, E_i\,, \tag{14}$$

for some integers $a_i > 0$, where the E_i's are the exceptional divisors of the blow-up $\tilde{S} \to S$. Then,

$$\begin{array}{lcl} \chi(\mathcal{O}_X) & = & 2(1 - g(C)) + \frac{1}{2}(L^2 + K_{\tilde{S}} L) \\ h^{01}(X) = h^1(\mathcal{O}_X) & = & g(C) + h^1(\mathcal{L}^\vee) \\ h^{02}(X) = p_g(X) & = & h^2(\mathcal{L}^\vee)\,. \end{array}$$

In particular, the Picard scheme of X is reduced if and only if $h^1(\mathcal{L}^\vee) = 0$.

PROOF. The formula for $\chi(\mathcal{O}_X)$ follows from (12) and Riemann-Roch on $\tilde{S}^{(-1)}$. The long exact sequence in cohomology applied to (12) together with the fact that $h^2(\mathcal{O}_{\tilde{S}^{(-1)}})$ vanishes (we assumed F to be rational), yields the formulae for h^{01} and h^{02}. The Albanese variety is $g(C)$-dimensional by Proposition 3.2. Thus, $\mathrm{Pic}(X)$ is reduced iff $h^1(\mathcal{O}_X) = g(C)$, i.e. iff $h^1(\mathcal{L}^\vee) = 0$.

To determine $\mathcal{L}$, we run through the arguments given in [E2, Section I.1]. The morphism ϖ is an $\alpha_{\mathcal{L}}$-torsor. This defines an embedding of $\mathcal{L}^{\otimes(-p)}$ into $\Omega^1_{\tilde{S}^{(-1)}}$. The annihilator $\mathcal{M} \subseteq \Theta^1_{\tilde{S}^{(-1)}}$ is the 1-foliation defining the morphism $\pi^{(-1)} : \tilde{S}^{(-1)} \to X^{(-1)}$ and we obtain a short exact sequence

$$0 \to \mathcal{M} \to \Theta^1_{\tilde{S}^{(-1)}} \to \mathcal{L}^{\otimes p} \to 0\,. \tag{15}$$

The line bundle $\mathcal{M}$ is isomorphic to $\Omega^\vee_{\tilde{S}^{(-1)}/X^{(-1)}}$. Taking determinants we obtain for $p = 2$ and together with (10) and (11) an equality

$$\mathcal{L}^{\otimes(-2)} \cong \omega_{\tilde{S}^{(-1)}} \otimes \mathcal{O}_{\tilde{S}^{(-1)}}((\tilde{\delta}))\,. \tag{16}$$

The divisor of δ is given by (8). Resolving the singularities of δ via a blow-up $\tilde{S} \to S$, the divisor $\tilde{\delta}$ of the pull-back of δ is given by

$$(\tilde{\delta}) = -(\delta_C)_\infty \cdot F - (\delta_F)_\infty \cdot C + \sum_i b_i E_i$$

for some non-negative integers b_i as explained in the proof of [Hi, Proposition 2.6]. From this and (16), formula (14) follows immediately. $\square$

Proposition 4.2. *Under the assumptions and notations of Proposition 4.1 we have*

$$h^{10}(X) = h^0(\Omega^1_X) = h^0(\tilde{S}, \varpi_*\varpi^*\mathcal{O}_{\tilde{S}^{(-1)}}(-\tilde{\delta})) \geq g(C)\,.$$

There exists a short exact sequence

$$0 \to H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(-\tilde{\delta})) \to H^0(X, \Omega^1_X) \stackrel{d_X}{\to} H^0(X, \Omega^2_X)\,,$$

where d_X denotes the differential.

PROOF. We rewrite the exact sequence (10) as

$$0 \to F^*\sigma^*\Omega_{X/\tilde{S}^{(-1)}} \stackrel{\beta}{\to} \varpi^*\Omega^1_{\tilde{S}^{(-1)}} \to \Omega^1_X \to \Omega_{X/\tilde{S}^{(-1)}} \to 0\,. \tag{17}$$

From (11) and (13) we obtain $\Omega_{X/\tilde{S}^{(-1)}} \cong \varpi^*(\mathcal{L}^\vee)$. Using (12), we see that $\varpi_*\Omega_{X/\tilde{S}^{(-1)}}$ is an extension of $\mathcal{L}^{\otimes(-2)}$ by $\mathcal{L}^\vee$. It is not difficult to see that neither of these line bundles has global sections. Hence $\Omega_{X/\tilde{S}^{(-1)}}$ has no non-trivial global sections.

We take determinants in (17) and use (15) as well as (16) to conclude that the cokernel of β is the pullback $\varpi^*\mathcal{E}$ of the cokernel $\mathcal{E}$ of

$$0 \to \mathcal{L}^{\otimes(-2)} \to \Omega^1_{\tilde{S}^{(-1)}} \to \mathcal{O}_{\tilde{S}^{(-1)}}(-\tilde{\delta}) \to 0\,. \tag{18}$$

By what we have already shown, we know that $h^0(X, \Omega^1_X) = h^0(X, \varpi^*\mathcal{E}) = h^0(\tilde{S}, \varpi_*\varpi^*\mathcal{E})$. By the projection formula and (12), $\varpi_*\varpi^*\mathcal{E}$ is an extension

$$0 \to \mathcal{O}_{\tilde{S}^{(-1)}}(-\tilde{\delta}) \to \varpi_*\varpi^*\mathcal{E} \to \mathcal{O}_{\tilde{S}^{(-1)}}(-\tilde{\delta}) \otimes \mathcal{L}^\vee \to 0\,.$$

By (13), the pull-back ϖ^* of the cokernel is isomorphic to ω_X. That the induced morphism on global sections from $H^0(X, \Omega^1_X) \cong H^0(\tilde{S}^{(-1)}, \varpi_*\varpi^*\mathcal{E})$ to $H^0(\tilde{S}^{(-1)}, \mathcal{E} \otimes \mathcal{L}^\vee) \cong H^0(X, \omega_X)$ coincides with the differential map follows from [Hi, Theorem 3.1] applied to $\tilde{S}$ and $\tilde{\delta}$. $\square$

Rational singularities. The cohomology of line bundles on $C \times F$ is easily computed. However, in order to obtain the Hodge invariants of a surface X constructed from data (C, F, δ), we have to desingularise $X' \to (C \times F)^{(-1)}$ and in order to use Proposition 4.1 and Proposition 4.2 we have to compute the cohomology of line bundles on a blow-up of $(C \times F)^{(-1)}$, which is usually more complicated. Things become easier if we assume that X' has at worst rational singularities.

A non-trivial vector field δ_C on a curve of genus $g(C)$ has $d_C := \deg(\delta_C)_\infty \geq 2 - 2g(C)$. The Hodge invariants become easier to compute if we assume that equality does not hold:

Proposition 4.3. *Let X be a uniruled surface constructed from data (C, F, δ). Assume that the singular quotient $X' := S/\delta$ has at worst rational singularities and $d_C > 2g(C) - 2$. Then*
 (1) $h^1(X,\, \mathcal{O}_X) \,=\, g(C)$ *and*
 (2) $h^0(X,\, \Omega^1_X) \,=\, g(C)$.

In particular, the Picard scheme of X is reduced, all global 1-forms on X are pull-backs of global 1-forms from $\mathrm{Alb}(X)$ and all global 1-forms on X are d-closed.

PROOF. We consider the finite flat morphism $\varrho : X' \to S^{(-1)}$. We define a line bundle $\mathcal{N}$ on $S^{(-1)}$ associated to ϱ as in (12). Then $\mathcal{N}^\vee$ is numerically equivalent to

$$\mathcal{N}^\vee \;\equiv\; \mathcal{O}_{S^{(-1)}}\left((-\frac{1}{2}d_F - 1)C \,+\, (-\frac{1}{2}d_C + g(C) - 1)F\right) .$$

We assumed $d_C > 2g(C) - 2$ and so the Künneth formula yields $h^1(\mathcal{N}^{\otimes(-i)}) = 0$ for $i \geq 1$.

Since X' has at worst rational singularities, we have an equality $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_{X'}) = g(C) + h^1(S^{(-1)}, \mathcal{N}^\vee)$, which is equal to $g(C)$ by the previous paragraph. Hence the Picard scheme of X is reduced.

Since X' is Gorenstein and we assumed that it has rational singularities it assumption it has Du Val singularities only. On the other hand, the resolution of singularities $p : X \to X'$ is minimal by [S-B2, Lemma 2.1] and we obtain $p^*\omega_{X'} \cong \omega_X$. For the dualising sheaves of X and X' we have the formulae $\omega_X \cong \varpi^*(\omega_{\tilde{S}} \otimes \mathcal{L})$ and $\omega_{X'} \cong \varrho^*(\omega_S \otimes \mathcal{N})$. From this we deduce an isomorphism

$$\mathcal{L} \;\;\cong\;\; f^*(\mathcal{N} \otimes \omega_S) \,\otimes\, \omega_{\tilde{S}}^\vee\,, \tag{19}$$

where $f : \tilde{S} \to S$ denotes the blow-up needed to resolve the isolated singularities of δ.

By Proposition 4.2, the global sections of $\mathcal{O}_{\tilde{S}^{(-1)}}(-\tilde{\delta})$ correspond to the global d-closed 1-forms on X. Pushing forward (18) to $S^{(-1)}$, we obtain a long exact sequence

$$\begin{array}{l} 0 \;\to\;\; \mathcal{N}^{(-2)} \;\to\; \Omega^1_{S^{(-1)}} \;\to\; f_*\mathcal{O}_{\tilde{S}^{(-1)}}(-\tilde{\delta}) \;\to \\ \qquad R^1 f_*\mathcal{N}^{(-2)} \;\to\; R^1 f_*\Omega^1_{S^{(-1)}} \;\to\; R^1 f_*\mathcal{O}_{\tilde{S}^{(-1)}}(-\tilde{\delta}) \;\to\; 0\,. \end{array}$$

It is not difficult to see that $R^1 f_*\mathcal{O}_{\tilde{S}^{(-1)}}(-\tilde{\delta}) \,=\, 0$. If $f \,:\, \tilde{S} \,\to\, S$ is a sequence of r blow-ups along closed points then by an induction on the number of blow-ups and using (19) we conclude that $R^1 f_*\mathcal{N}^{(-2)}$ and $R^1 f_*\Omega^1_{S^{(-1)}}$ both are Artin algebras of length r. In particular, we obtain a short exact sequence

$$0 \,\to\, \mathcal{N}^{(-2)} \,\to\, \Omega^1_{S^{(-1)}} \,\to\, f_*\mathcal{O}_{\tilde{S}^{(-1)}}(-\tilde{\delta}) \,\to\, 0\,. \tag{20}$$

Taking cohomology and noting that $H^0(\mathcal{N}^{(-2)}) = H^1(\mathcal{N}^{(-2)}) = 0$ we see that the space of d-closed global 1-forms on X is $g(C)$-dimensional.

Let x, y be local coordinates on $\tilde{S}^{(-1)}$ such that $y = 0$ defines a component E' of the exceptional divisor of the blow-up f. We also assume that E' is the only possible component of the divisor of $\tilde{\delta}$ through the point $(x, y) = (0, 0)$. We denote by e the order of pole $\mathcal{O}_{\tilde{S}^{(-1)}}(-\tilde{\delta})$ has along E', a number

which can be read off from (19). If E' is an integral curve for the foliation $\tilde{\delta}$ then $\varpi^*\mathcal{O}_{\tilde{S}^{(-1)}}(-\tilde{\delta})$ is locally generated by $(\varpi^* y)^e\, d(\varpi^* y)$ as a subsheaf of Ω^1_X. Then, $\varpi^* y$ and a square root $\sqrt{\varpi^* x}$ form a system of local coordinates on X. If E' is not an integral curve for $\tilde{\delta}$ then $\varpi^*\mathcal{O}_{\tilde{S}^{(-1)}}(-\tilde{\delta})$ is locally generated by $(\varpi^* y)^e\, d(\varpi^* x)$. In this case, $\varpi^* x$ and a square root $\sqrt{\varpi^* y}$ form a system of local coordinates on X.

We have already seen in the proof of Proposition 4.2 that all global 1-forms are global sections of $\varpi^*\mathcal{O}_{\tilde{S}^{(-1)}}(-\tilde{\delta})$. Whence the derivative of a global 1-form is can be computed on forms of the form a regular function times $(\varpi^* y)^e\, d(\varpi^* x)$ and $(\varpi^* y)^e\, d(\varpi^* y)$, respectively. Using the local coordinates on X above, we see that differentiating such a 1-form we get a 2-form, which has a zero of order at least e along $\varpi^*(y) = 0$. On the other hand, X' has Du Val singularities only, which have no adjunction condition, i.e. Ω^2_X is locally around $(x, y) = (0, 0)$ generated by $d(\varpi^* y) \wedge d\sqrt{\varpi^* x}$ if E' is an integral curve and $d(\sqrt{\varpi^* y}) \wedge d(\varpi^* x)$ if E' is not an integral curve for $\tilde{\delta}$.

We define $\mathcal{F} := \omega_{S^{(-1)}} \otimes \omega^\vee_{\tilde{S}^{(-1)}}$, which we consider as subsheaf of $\mathcal{O}_{\tilde{S}^{(-1)}}$. By (19) and our local computations, we see that $d\Omega^1_X$ is a subsheaf of $\varpi^{-1}(\mathcal{F}) \cdot \Omega^2_X \;=\; \varpi^*\mathcal{F} \otimes \Omega^2_X$. Taking global sections we get

$$H^0(\Omega^1_X) \overset{d_X}{\to} H^0(\varpi^*(\mathcal{F}) \,\otimes\, \Omega^2_X) \;\subseteq\; H^0(\Omega^2_X)\,.$$

In order to show that all global 1-forms are d-closed it is enough to show that the dimension of the space in the middle is zero. Pushing the sheaf forward to $S^{(-1)}$ and using (13) we obtain an extension

$$0 \,\to\, \mathcal{F} \otimes (\omega_{\tilde{S}^{(-1)}} \otimes \mathcal{L}) \,\to\, \varpi_*(\varpi^*(\mathcal{F}) \,\otimes\, \Omega^2_X) \,\to\, \mathcal{F} \otimes \omega_{\tilde{S}^{(-1)}} \,\to\, 0\,.$$

The sheaf on the right has only trivial global sections and so we have to compute those on the left. With (20) we obtain a short exact sequence

$$0 \,\to\, \mathcal{N}^{(-3)} \,\to\, \Omega^1_{S^{(-1)}} \otimes \mathcal{N}^\vee \,\to\, f_*\left(\mathcal{F} \,\otimes\, (\omega_{\tilde{S}^{(-1)}} \otimes \mathcal{L})\right) \,\to\, 0\,. \tag{21}$$

Using the vanishing of $H^1(\mathcal{N}^{(-3)})$ and $H^0(\Omega^1_S \otimes \mathcal{N}^\vee)$, it finally follows that all global 1-forms on X are d-closed.

The Albanese variety of X is $g(C)$-dimensional by Proposition 3.2. Igusa's theorem [Ig] states that the pull-back of a non-trivial global 1-form on $\mathrm{Alb}(X)$ to X remains non-trivial. This implies $h^0(\Omega^1_X) \geq g(C)$. If equality holds then every global 1-from on X is the pull-back of a global 1-form on $\mathrm{Alb}(X)$. $\square$

Remark 4.4. Being Gorenstein and having rational singularities, X' has at worst Du Val singularities. It is not difficult to see that these can be of type A_1, D_{2n}, E_7 and E_8 only, cf. [S-B2, Lemma 2.1].

The Frölicher spectral sequence. For a smooth variety X, Hodge and de Rham cohomology are related by the so-called *Frölicher spectral sequence*

$$E_1^{ij} \;:=\; H^j(X,\, \Omega^i_{X/k}) \;\Rightarrow\; H^{i+j}_{\mathrm{dR}}(X/k).$$

It follows from classical Hodge theory that this spectral sequence degenerates at E_1-level for Kähler manifolds. For curves and complex surfaces we even have degeneration at E_1-level without the Kähler assumption, cf. [BHPV, Chapter IV.2]. However, for surfaces over arbitrary fields, there is no reason for this spectral sequence to degenerate at E_1-level.

Theorem 4.5. *Let X be a uniruled surface constructed from data (C, F, δ) in characteristic 2 such that $(C \times F)/\delta$ has at worst rational singularities. Then the crystalline cohomology of X is torsion-free and its Frölicher spectral sequence degenerates at E_1-level.*

PROOF. Since $H^1_{\mathrm{cris}}(X/W)$ is a torsion-free W-module, we know from Proposition 3.2 that its rank is equal to $2g := 2g(C)$.

We consider the universal coefficient formula

$$0 \,\to\, H^i_{\mathrm{cris}}(X/W)\otimes_W k \,\to\, H^i_{\mathrm{dR}}(X/k) \,\to\, \mathrm{Tor}_1^W(H^{i+1}_{\mathrm{cris}}(X/W),\, k) \,\to\, 0\,. \tag{22}$$

Thus, $H^1_{\mathrm{dR}}(X/k)$ is at least $2g$-dimensional. The existence of the Frölicher spectral sequence already implies the inequality $h^1_{\mathrm{dR}} \leq h^{01}+h^{10}$. Since $h^{01}=h^{10}$ by Proposition 4.1 and Proposition 4.3 we have equality. Because $H^4_{\mathrm{cris}}(X/W)$ is torsion-free, the universal coefficient formula (22) for $i=3$ yields $h^3_{\mathrm{dR}}=2g$. By Serre duality, we obtain $h^3_{\mathrm{dR}}=2g=h^{01}+h^{10}=h^{12}+h^{21}$. Again, it follows already from the existence of the Frölicher spectral sequence that the sum over the $(-1)^i h^i_{\mathrm{dR}}$ is equal to the sum $(-1)^{i+j}h^{ij}$. Since we are working with surfaces, this implies $h^2_{\mathrm{dR}}=h^{02}+h^{11}+h^{20}$. Hence h^n_{dR} is equal to the sum over all h^{ij} with $i+j=n$ for all n. This implies that the Frölicher spectral sequence degenerates at E_1-level.

Plugging this into (22), we see that $\mathrm{Tor}_1^W(H^i_{\mathrm{cris}}(X/W),\, k)=0$ for all i which implies that the crystalline cohomology of X is torsion-free. $\square$

The slope spectral sequence. In [Ill, Section II.3], Illusie constructs a a spectral sequence from Hodge-Witt cohomology to crystalline cohomology

$$E_1^{ij} \,:=\, H^j(X,\, W\Omega^i_X) \,\Rightarrow\, H^{i+j}_{\mathrm{cris}}(X/W).$$

Modulo torsion, this sequence always degenerates at E_1-level, cf. [Ill, Théorème II.3.2]. In general, degeneracy at E_1-level is equivalent to the torsion subgroups of the $H^j(W\Omega^i_X)$'s being finitely generated W-modules. For surfaces, Nygaard has shown that this is equivalent to the finite generation of $H^2(W\mathcal{O}_X)$, cf. [Ill, Corollaire II.3.14].

If the slope spectral sequence degenerates at E_1-level then the crystalline cohomology decomposes as a direct sum of the Hodge-Witt cohomology groups and the variety is said to be of *Hodge-Witt type*.

Theorem 4.6. *Let X be a uniruled surface constructed from data (C,F,δ) in characteristic 2 such that $(C\times F)/\delta$ has at worst rational singularities. If $\chi(\mathcal{O}_X)>1-g(C)$ then the slope spectral sequence does not degenerate at E_1-level.*

PROOF. By the previous result we know that the crystalline cohomology of X is torsion-free. Since $\tilde{S}$ is birationally ruled, $H^2_{\mathrm{cris}}(X/W)\otimes_W K$ is pure of slope 1 and so $H^2_{\mathrm{cris}}(X/W)\otimes_W K$ is pure of slope 1. Being torsion-free, already $H^2_{\mathrm{cris}}(X/W)$ is pure of slope 1.

Suppose that the slope spectral sequence degenerates. Then we have a Hodge-Witt decomposition of the crystalline cohomology of X, cf. [IR, Théorème IV.4.5]. In particular, $H^2(W\mathcal{O}_X)$ can be identified with the part of $H^2_{\mathrm{cris}}(X/W)$ that has slope strictly less than 1. Since $H^2_{\mathrm{cris}}(X/W)$ is pure of slope 1, we see that $H^2(W\mathcal{O}_X)$ is zero.

For every $n\geq 1$, the Verschiebung V induces a short exact sequence

$$0 \,\to\, VW_{n-1}\mathcal{O}_X \,\to\, W_n\mathcal{O}_X \,\to\, \mathcal{O}_X \,\to\, 0\,.$$

Taking cohomology, passing to the inverse limit and noting that $H^3(VW_{n-1}\mathcal{O}_X)$ vanishes for all n, we obtain a surjective homomorphism of W-modules from $H^2(W\mathcal{O}_X)$ onto $H^2(\mathcal{O}_X)$.

However, $H^2(\mathcal{O}_X)\neq 0$ since we assumed $\chi(\mathcal{O}_X)>1-g$ whereas $H^2(W\mathcal{O}_X)$ is zero. This contradiction shows that the slope spectral sequence does not degenerate at E_1-level. $\square$

For a smooth variety X the sheaf $B\Omega^i_X$ is defined to be the image of $d:\Omega^{i-1}_X\to\Omega^i_X$. Then X is called *ordinary* if $H^j(X,B\Omega^i_X)=0$ for all i,j. By [IR, Théorème IV.4.13], the slope spectral sequence of an ordinary variety degenerates at E_1 and thus we obtain

Corollary 4.7. *A uniruled surface constructed from data (C, F, δ) as in Theorem 4.6 is not ordinary.* $\square$

Remark 4.8. If a surface X is fibred over some curve C such that the generic fibre is a *smooth* rational curve, then X is birationally ruled and such a surface is ordinary if and only if the curve C is ordinary.

5. ARITHMETIC OBSERVATIONS

Artin invariants. A basic invariant of a surface is its Picard number ρ, i.e. the rank of its Néron–Severi group. By the Igusa–Severi inequality, we always have $\rho \leq b_2$ and a surface is called *supersingular in the sense of Shioda* if equality holds. When Artin studied supersingular K3 surfaces [Ar], he observed that the discriminant of the intersection form on the Néron–Severi group is always negative and an even power of the characteristic of the ground field.

Proposition 5.1. *Let X be a uniruled surface constructed from data (C, F, δ).*

Then X is supersingular in the sense of Shioda. The discriminant of its Néron–Severi lattice is

$$\operatorname{disc} \mathrm{NS}(X) \;=\; -\,p^{2\sigma}\,,$$

where σ is a non-negative integer, i.e. there exists an Artin invariant σ for such surfaces.

It is bounded by $\sigma \leq b_2/2$. If the crystalline cohomology of X is torsion-free then $\sigma \geq p_g$.

PROOF. There exists a finite purely inseparable map π from a birationally ruled surface $\tilde{S}$ onto X, where S is ruled over the curve C. The intersection form on $NS(\tilde{S})$ is unimodular. By the Hodge index theorem, its signature is equal to $(1, \rho(\tilde{S}) - 1)$. Hence its discriminant is equal to $(-1)^{\rho(\tilde{S})-1}$.

Since π is inseparable, $\pi^* NS(X)$ is a subgroup of finite index in $NS(\tilde{S})$. In particular, we have $\rho(X) = \rho(\tilde{S}) =: \rho$, i.e. X is supersingular in the sense of Shioda. As π has degree p, this index is equal to p^n for some integer n. Hence the discriminant of $\pi^* NS(X)$ is equal to $(-1)^{\rho-1} p^{2n}$. Using the projection formula $\pi^* A \cdot \pi^* B = pA \cdot B$, it follows that the discriminant of $NS(X)$ is equal to $(-1)^{\rho-1} p^{2n-\rho}$. So once we have shown that $2n - \rho$ is even it follows that ρ is even and that the sign of the discriminant is -1.

The first Chern class provides us with an injective homomorphism

$$c_1 \;:\; NS(X) \otimes W \hookrightarrow H^2_{\mathrm{cris}}(X/W)$$

where the intersection pairing on the left hand side coincides with the cup-product on the right hand side, cf. [Ill, Remarque II.5.21.4]. By Poincaré duality, the cup-product on $H^2_{\mathrm{cris}}(X/W)$ (modulo torsion) is unimodular. Both sides are free of rank $\rho = b_2$ and so the image of $c_1(NS(X))$ in $H^2_{\mathrm{cris}}(X/W)$ has finite index. This index is a p-power and it follows that the discriminant of $NS(X) \otimes W$ is an even p-power.

Let $F : \tilde{S} \to \tilde{S}^{(-1)}$ be the Frobenius morphism. Then $F^*(NS(\tilde{S}^{(-1)}))$ is a sublattice of $\pi^* NS(X)$ since F factors over π. The index of $F^*(NS(\tilde{S}^{(-1)}))$ in $NS(\tilde{S})$ is equal to p^ρ. Thus, the index p^n of $\pi^* NS(X)$ in $NS(\tilde{S})$ divides p^ρ. This implies $n \leq \rho$, and so $2\sigma = 2n - \rho \leq \rho = b_2$.

If the crystalline cohomology of X is torsion-free then $\mathrm{Pic}(X)$ is reduced by [Ill, Proposition II.5.16]. Then we can argue as in [Ill, Remarque II.5.21] to conclude $\sigma \geq p_g(X)$. $\square$

It is known that the Néron–Severi lattice of a $K3$ surface is even and unimodular. More precisely, it is a sublattice of $3H \oplus (-2)E_8$, where H is a hyperbolic plane. By a result of Rudakov and Šafarevič, the Artin invariant determines the intersection form of a supersingular K3 surface up to isomorphism. In our case, the situation is more complicated. In some examples, we have to resolve an elliptic $(19)_0$-singularity to obtain our surface. Then there is a curve with self-intersection -3 and so the intersection form is not even.

The Artin–Tate conjecture. Let X be a surface over the finite field k with q elements and let $Z(X,t)$ be its zeta function. By Deligne's proof of the Weil conjectures it is known that there exist polynomials P_1, P_2 and P_3 of degrees equal to the Betti numbers b_1, b_2 and b_3 of X such that

$$Z(X,\,t)\;=\;\frac{P_1(X,\,t)\,\cdot\,P_3(X,\,t)}{(1-q\,t)\,\cdot\,P_2(X,\,t)\,\cdot\,(1-q^2\,t)}\,.$$

We denote by $\rho(X)$ the Picard number and by $\mathrm{Br}(X)$ the Brauer group of X. We define $\alpha(X):=\chi(\mathcal{O}_X)-1+b_1(X)/2$. The Artin–Tate conjecture states that

$$P_2(X,\,q^{-s})\;\sim\;(-1)^{\rho(X)-1}\,\cdot\,\frac{\operatorname{disc}\mathrm{NS}(X)\,\cdot\,|\mathrm{Br}(X)|}{|NS(X)_{\mathrm{tors}}|\,\cdot\,q^{\alpha(X)}}\,\cdot\,(1-q^{1-s})^{\rho(X)}\text{ as }s\to 1\,.$$

Proposition 5.2. *Let X be a surface constructed from data (C,F,δ), where C, F and δ are defined over the finite field k with q elements.*

Then X is defined over k, the Artin–Tate conjecture holds for X, and we have an equality

$$\operatorname{disc}\mathrm{NS}(X)\,\cdot\,|\mathrm{Br}(X)|\;=\;q^{\alpha(X)\,-\,g(C)\cdot g(F)}\,\cdot\,|NS(X)_{\mathrm{tors}}|\,.$$

PROOF. To construct X we had to find a blow-up $\tilde{S}$ of $S=C\times F$ and quotiented by a vector field to obtain $\pi:\tilde{S}\to X$. All this can be done over k and so X is defined over k.

Since the Tate conjecture holds for S, it also holds for $\tilde{S}$, cf. the introduction of [Mi]. Then the Tate conjecture also holds for X since there is a finite morphism from a surface for which the Tate conjecture holds onto X, namely π. But the truth of the Tate conjecture implies the truth of the Artin–Tate conjecture by the main result of [Mi]. According to [LLR], the results of [Mi] are also true in characteristic $p=2$.

Since π is purely inseparable, the zeta functions of $\tilde{S}$ and X coincide. In particular, we have $P_2(X,t)=P_2(S,t)$. Together with the fact that $\mathrm{Br}(\tilde{S})$ is trivial and $\alpha(\tilde{S})=g(C)\cdot g(F)$, we obtain the stated formula for $\operatorname{disc}\mathrm{NS}(X)$. □

6. EXAMPLES

Bogomolov–Miyaoka–Yau. For a surface over the complex numbers, this inequality states $c_1^2\leq 9\chi$. It is known to fail in positive characteristic, see e.g. [Sz, Section 3.4.1] or [BHH, Kapitel 3.4.J]. Over the complex numbers, $\chi=1$ is the lowest value possible for a surface of general type. These surfaces have been studied for quite some time and so it is interesting to note that also among them there are counter-examples to this inequality in positive characteristic:

Theorem 6.1. *There exist surfaces of general type with $\chi=1$ and $c_1^2=14$ in characteristic 2.*

PROOF. We let $f:C\to\mathbb{P}^1$ be an Artin–Schreier curve of genus 3 as in Section 1. As rational vector field δ_C we choose the additive vector field $\partial/\partial x$ from (6), which has a pole of order 4 at infinity and no zeros. On $F:=\mathbb{P}^1$ we choose the vector field $\delta_F:=\delta_1$ from (1), which has a pole of order 4 and three zeros of order 2.

The quotient $(C\times F)/(\delta_C+\delta_F)$ has exactly one singularity, which is elliptic of type $(19)_0$. The resolution of singularities yields a surface of general type with $\chi=1$ and $c_1^2=14$. □

Remark 6.2. By Noether's formula $12\chi=c_1^2+c_2$, this surface has negative c_2. Using an Artin–Schreier curve of genus 4 curve instead of genus 3 in the previous construction, we obtain a surface of general type with $\chi=3$ and $c_1^2=30$, i.e. a counter-example to the Bogomolov–Miyaoka–Yau inequality with small χ and c_2 positive.

Bounds on (-2)-curves. Over the complex numbers, a theorem of Miyaoka bounds the number of disjoint (-2)-curves on a minimal surface of general type above by $\frac{1}{9}(3c_2 - c_1^2)$, cf. [BHPV, Section VII.4]. This may fail in in positive characteristic and Shepherd-Barron [S-B2, Theorem 4.1] has shown that if there exist more than $c_1^2 + \frac{1}{2}c_2$ disjoint (-2)-curves then the surface in question is uniruled.

However, there is usually a gap between these two bounds and the following theorem shows us that there exist uniruled as well as non-uniruled surfaces in this gap.

Theorem 6.3. *There exist minimal surfaces of general type in characteristic* 2 *that violate Miyaoka's bound on* (-2)*-curves that do not reach Shepherd-Barron's bound. There exist uniruled as well as non-uniruled such surfaces.*

The non-uniruled surface that we present is birationally dominated by an Abelian surface.

PROOF. First, we give a non-uniruled example. Let $E := E_\alpha$ be an elliptic curve as in Section 1 with $\alpha \neq 0$, i.e. E is not supersingular. The vector field $\delta_E := \delta_{\alpha,1,\alpha}$ defined by (4) is an additive rational vector field on E with a zero and a pole of order 2.

We apply our construction to the data $(E, E, \delta_E + \delta_E)$. The singular quotient $(E \times E)/(\delta_E + \delta_E)$ has two singularities of type D_4.

Resolving the singularities we obtain a minimal surface X of general type with $\chi = 1$, $c_1^2 = 4$ and hence $c_2 = 8$. There are 6 isolated (-2)-curves coming from the two D_4-singularities and hence Miyaoka's bound is violated.

By construction, this surface is inseparably dominated by a blow-up of the Abelian surface $E \times E$. Factoring the Frobenius morphism and blowing down the (-1)-curves, we obtain a surjective morphism from X onto the Abelian surface $(E \times E)^{(-1)}$. If X were uniruled, we would have a dominant map $Y \to X$ from some birationally ruled surface Y. Thus, there would exist a dominant map from Y to an Abelian surface which would have to factor over the Albanese map of Y. This is absurd since the image of the Albanese map of Y is a curve. Hence, X is not uniruled.

To obtain a uniruled example, we do the computations with the example of Theorem 6.1 but with an Artin–Schreier curve of genus 2 instead of genus 3. The singular quotient has exactly one singularity, which is of type D_8. The resolution is a surface X with $\chi = 1$, $c_1^2 = 8$ and $c_2 = 4$. There are 5 disjoint (-2)-curves coming from the resolution of the D_8-singularity which is already enough to violate Miyaoka's bound. On the other hand, this surface has $b_2 = 10$ and so the rank of its Néron–Severi group is at most 10 by the Igusa–Severi inequality. Disjoint (-2)-curves are linearly independent in the Néron–Severi group and they span a negative definite lattice. By the Hodge index theorem, there can be at most 9 disjoint (-2)-curves on X. Hence Shepherd-Barron's bound is not reached. □

7. UNBOUNDED PATHOLOGICAL BEHAVIOUR

Nonreduced Picard schemes. In characteristic zero, Cartier's theorem states that group schemes over a field are smooth. This may fail in positive characteristic, and so one has to distinguish between the Picard scheme and the Picard variety of a given variety. A first example of a smooth variety with non-reduced Picard scheme was found by Igusa [Ig2]. Whereas $\frac{1}{2}b_1$ gives the dimension of the Picard variety, the number h^{01} gives the dimension of the tangent space to the Picard scheme and hence we only have an inequality $\frac{1}{2}b_1 \leq h^{01}$, with equality if and only if the Picard scheme is reduced.

Theorem 7.1. *Given an integer $q \geq 2$, there exists a family $\{X_i\}_{i\in\mathbb{N}}$ of uniruled surfaces of general type in characteristic 2 all having the same Picard variety of dimension q such that*

$$h^{01}(X_i) \,=\, h^1(\mathcal{O}_{X_i}) \,\to\, \infty \text{ as } i \to \infty\,.$$

Thus, the Picard scheme can get arbitrarily non-reduced, even when fixing the Picard variety.

PROOF. We let $h := q+1$ and let $\varphi : C \to \mathbb{P}^1$ be the Artin–Schreier cover of $\mathbb{P}^1$ given by $z^2 - z = x^{2h-1}$, which defines a curve of genus q as explained in Section 1. Let m be the largest integer less or equal to $\frac{1}{2}(q-1)$. We choose m distinct elements $\{a_i\}_{i=1,\dots,m}$ of the ground field k and consider the rational vector field

$$\delta_C \,:=\, \varphi^* \left(\prod_{i=1}^m \frac{1}{(x-a_i)^2} \cdot \frac{\partial}{\partial x} \right)\,.$$

This rational vector field is additive and has $(g-1)$ poles of order 2 on C.

We let $F := \mathbb{P}^1$ and choose additive vector fields with poles and zeros of order 2. Clearly, we can find a family $\{\delta_F^i\}_{i\in\mathbb{N}}$ of such vector fields such that the degree d_F^i of its divisor of poles tends to infinity. For example, we can use vector fields as in (3).

We set $\delta^i := \delta_C + \delta_F^i$ on $S := C \times F$ and let X_i be the uniruled surface constructed from data (C, F, δ^i). By Proposition 3.2, the Albanese variety, i.e. the dual of the Picard variety, of the X_i's is the Jacobian of $C^{(-1)}$.

Since δ_C and δ_F^i have only poles of order 2, the singular quotient $X_i' := S/\delta^i$ is a normal surface, which has only Du Val singularities of type D_4. In particular, we can compute $h^{01}(X_i)$ on X_i'. By Proposition 4.1, this number equals $g(C) + h^1(\mathcal{N}_i^\vee)$, where $\mathcal{N}_i$ is given by

$$\mathcal{N}_i^{\otimes(-2)} \,\cong\, \mathcal{O}_F(-d_F^i - 2) \boxtimes \mathcal{O}_C$$

In Section 1 we noted that the 2-rank of the Artin–Schreier curve C is zero, which implies that

$$\mathcal{N}_i^\vee \,\cong\, \mathcal{O}_F(-\frac{d_F^i}{2} - 1) \boxtimes \mathcal{O}_C\,.$$

Using the Künneth formula, we see that $h^1(\mathcal{N}_i^\vee)$ tends to infinity as d_F^i tends to infinity. □

Global 1-forms. In characteristic zero, Hodge theory implies that every global 1-form on a variety is the pull-back of a 1-form from its Albanese variety via the Albanese map. Igusa [Ig] showed that the pull-back of a non-trivial global 1-form from the Albanese variety of a smooth variety via the Albanese map remains non-trivial in arbitrary characteristic. Therefore, one always has the inequality $h^{10}(X) = h^0(\Omega_X^1) \geq h^0(\Omega^1_{\mathrm{Alb}(X)}) = \frac{1}{2}b_1(X)$. On the other hand, Igusa [Ig2] also gave an example of a surface in positive characteristic with strict inequality.

Theorem 7.2. *Given an integer $q \geq 2$, there exists a family $\{X_i\}_{i\in\mathbb{N}}$ of uniruled surfaces of general type in characteristic 2 all having the same Albanese variety of dimension q such that*

$$h^{10}(X_i) \,=\, h^0(\Omega^1_{X_i}) \,\to\, \infty \text{ as } i \to \infty\,.$$

PROOF. We take the family of surfaces from Theorem 7.1. The computations from the proof and the long exact sequence of cohomology applied to (20) yield $h^{10}(X_i) \geq g(C) + 3d_F^i/2 - 1$, which tends to infinity as i tends to infinity. □

Hodge symmetries. In characteristic zero one has not only that the Frölicher spectral sequence of a projective variety always degenerates at E_1-level but also the Hodge symmetries $h^{ij} = h^{ji}$. In positive characteristic, these symmetries can be violated even if the Frölicher spectral sequence degenerates at E_1-level, cf. [Se, Proposition 16] and [DI, Remarque 2.6 (ii)].

Theorem 7.3. *Given an integer $q \geq 2$, there exists a family $\{X_i\}_{i\in\mathbb{N}}$ of uniruled surfaces of general type in characteristic 2 all having the same Albanese variety of dimension q such that*

$$h^{10}(X_i) - h^{01}(X_i) \;\to\; \infty \;\text{ as }\; i \to \infty\,,$$

i.e. the Hodge symmetries fail.

PROOF. We use the family used in the proofs of Theorem 7.1 and Theorem 7.2, where we have seen $h^{01}(X_i) = g(C) + d_F^i/2$ and $h^{10}(X_i) \geq g(C) + d_F^i - 1$. □

Global vector fields. Over the complex numbers, a surface of general type has no non-trivial global vector fields. This follows from the fact that the vector space of global vector fields can be identified with the tangent space to the group of biholomorphic automorphisms, and this group is finite for a surface of general type. Counter-examples in positive characteristic can be found, e.g. in [La] or in [S-B2], where first examples of non-uniruled surfaces of general type with vector fields appear.

Proposition 7.4. *Let X be a surface constructed from data (C, F, δ) in characteristic 2. We assume that $g(C) \leq 1$, and $g(F) \leq 1$ as well as $p_g(X) \neq 0$. Then X possesses non-trivial global vector fields.*

PROOF. We keep the notations of the proof of Proposition 4.1. We consider the morphism $\varpi : X \to \tilde{S}^{(-1)}$. Dualising (10) and plugging in (11) we obtain an exact sequence

$$0 \;\to\; \omega_X \otimes \varpi^*\omega^\vee_{\tilde{S}^{(-1)}} \;\to\; \Theta^1_X \;\to\; \varpi^*\Theta^1_{\tilde{S}^{(-1)}} \;\to\; \ldots \tag{23}$$

The assumptions on $g(C)$, $g(F)$ and $p_g(X)$ make sure that ω_X and $\omega^\vee_{\tilde{S}^{(-1)}}$ have non-trivial global sections. Hence $\omega_X \otimes \varpi^*\omega^\vee_{\tilde{S}^{(-1)}}$ has non-trivial global sections, which yields non-trivial global vector fields via (23). □

Theorem 7.5. *In characteristic 2, there exist families of surfaces of general type where c_1^2 and χ tend to infinity and such that each member of this family possesses non-trivial global vector fields. Moreover, we can find such families in which every member is uniruled, resp. not uniruled.*

PROOF. First, we give unirational examples. We set $F := \mathbb{P}^1$ and choose a family of additive rational vector fields δ_F^i poles and zeros of order 2 such that d_F^i, the degree of the divisor of poles of δ_F^i, tends to infinity as i tends to infinity. For example, we could use vector fields of the form (3).

The quotient $X_i' := (F \times F)/(\delta_F^i + \delta_F^i)$ is a normal surface with Du Val singularities of type D_4. If X_i is the surface constructed from $(F, F, \delta_F^i + \delta_F^i)$ then its invariants χ and K^2 coincide with those of X_i'. Thus it is enough to show that $K^2_{X_i'}$ and $\chi(\mathcal{O}_{X_i'})$ are unbounded as i tends to infinity. However, this can easily be seen from Proposition 3.3 and Proposition 4.1.

By Proposition 7.4, the surfaces X_i possess non-trivial global vector fields.

To construct a family of surfaces of general type that is not uniruled, we let $\varphi : E \to \mathbb{P}^1$ be an elliptic curve given as a separable double cover branched over $x = 0$ and $x = \infty$. We choose $2n$ pairwise distinct and non-zero elements $\{a_i, b_i\}_{i=1,\ldots,n}$ of the ground field k and consider

$$\delta_E^n \;:=\; \varphi^*\left(\prod_{i=1}^{n}\frac{(x-a_i)^2}{(x-b_i)^2}\cdot\frac{\partial}{\partial x}\right)\,.$$

This defines an additive rational vector field on E with poles and zeros of order 2.

Arguing as before we see that the surfaces constructed from data $(E, E, \delta_E^n + \delta_E^n)$ have non-trivial global vector fields and that χ and c_1^2 tend to infinity as n tends to infinity. Arguing as in the proof of Theorem 6.3 we see that these surfaces are not uniruled. □

Inseparability of the canonical map. Curves of general type, i.e. of genus at least 2, fall into two classes: the hyperelliptic and the non-hyperelliptic ones. Whether a curve is hyperelliptic or not can be seen from the canonical map: it is either a separable morphism of degree 2 onto $\mathbb{P}^1$ or defines an embedding. In any case, it is a finite and separable morphism onto its image.

For surfaces, the canonical map may be empty and is usually only a rational map. The canonical map of a product of two curves of general type is a finite morphism onto a rational, a ruled or a general type surface, depending on whether the curves are hyperelliptic or not. In these examples, the canonical map is a separable morphism. We will see in the next theorem that the canonical map can become inseparable and that this is not a sporadic phenomenon.

It follows from Shepherd-Barrons work [S-B, Theorem 27] that for c_1^2 and χ sufficiently large, $|3K_X|$ defines a birational morphism. Also, if X has no pencil of curves of arithmetic genus 2 then already $|2K_X|$ defines a birational morphism if c_1^2 and χ are sufficiently large. However, the surfaces presented in Theorem 7.6 do not possess pencils of curves of small arithmetic genus. Hence the inseparability of the canonical map is not related to the existence of special fibrations of low genus.

Theorem 7.6. *In characteristic 2, there exist families of surfaces of general type where c_1^2 and χ tend to infinity and such that the canonical map of each member is a generically finite and inseparable morphism onto a rational surface.*

Moreover, given a natural number b, we can find such families that do not possess pencils of curves of arithmetic genus less than b.

PROOF. We consider the family of unirational surfaces of general type constructed in Theorem 7.5 and use the notations from the proof.

We will work on the singular surfaces X_i', which have only Du Val singularities of type D_4. The X_i's are $\alpha_{\mathcal{L}_i}$-torsors over $S^{(-1)} := (\mathbb{P}^1 \times \mathbb{P}^1)^{(-1)}$. It is not difficult to see that $\mathcal{L}_i$ and $\omega_{S^{(-1)}} \otimes \mathcal{L}_i$ are very ample line bundles on $S^{(-1)}$. From (13) it follows that the canonical sheaf on X_i' is ample. Hence, X_i' is the canonical model of X_i and the canonical map factors over X_i' which justifies to work with the X_i''s rather than the X_i's.

Let us analyse the canonical system of the X_i's. We have already seen above that there is a finite and purely inseparable morphism $\varpi_i : X_i' \to S^{(-1)}$ which has the structure of an $\alpha_{\mathcal{L}_i}$-torsor. Pushing $\omega_{X_i'}$ forward to $S^{(-1)}$ and using (13), we obtain an extension

$$0 \to \omega_{S^{(-1)}} \otimes \mathcal{L}_i \to \varpi_{i,*}(\omega_{X_i'}) \to \omega_{S^{(-1)}} \to 0\,.$$

The line bundle $\omega_{S^{(-1)}}$ has no global sections and it is not difficult to see that the line bundle $\omega_{S^{(-1)}} \otimes \mathcal{L}_i$ defines an embedding φ_i of $S^{(-1)}$ into $\mathbb{P}^{p_g(X_i')-1}$. Thus, the canonical map of X_i' factors as $\varphi_i \circ \varpi_i$. In particular, it is a finite and inseparable morphism onto a rational surface.

Suppose X_i has a pencil of curves of arithmetic genus d_i, say with generic fibre D_i. Then the adjunction formula yields $2d - 2 = K_{X_i} D_i$ since a fibre has $D_i^2 = 0$. The canonical divisor on X_i is the pull-back of a divisor on $\tilde{S}^{(-1)}$ by (13). Using the projection formula and (14), it is not difficult to see that if d_F^i is sufficiently large then $K_{X_i} D_i > 2b - 2$ for any given bound b. Hence, if d_F^i is sufficiently large, then X_i does not possess a pencil of curves of arithmetic genus less than b. □

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